

Quotient space: - The linear space L/M is called the Quotient space of L Modulo M . The mapping ϕ of L onto L/M defined by $\phi(x) = x+M$ is called the canonical mapping of L onto L/M .

Theorem: - Let M be a closed linear manifold (Subspace) in a normed linear space N . For each coset $x+M$ in the quotient space N/M we define

$$\|x+M\| = \inf \{ \|x+m\| : m \in M \}.$$

Then $\|x+M\|$ is a norm on N/M and thus N/M is a normed linear space. Further if N is a Banach space, then so is N/M .

Proof: - We verify all the postulates for a norm.

(n1): Since $\|x+m\|$ is a non-negative real number and every set of non-negative real numbers is bounded below, it follows that $\inf \{ \|x+m\| : m \in M \}$ exists and is non-negative, that is

$$\|x+M\| \geq 0 \quad \forall x+M \in N/M$$

(n2): Let $x+M = M$ (the zero element of N/M). Then $x \in M$.

$$\text{Hence } \|x+M\| = \inf \{ \|x+m\| : m \in M, x \in M \}.$$

$$= \inf \{ \|y\| : y \in M \} = 0$$

[$\because M$ being a subspace contains zero vector whose norm is real number 0]

Thus $x+M = M \Rightarrow \|x+M\| = 0$

Conversely, we have

$$\|x+M\| = 0 \Rightarrow \inf \{ \|x+m\| : m \in M \} = 0$$

\Rightarrow there exists a sequence $\{m_k\}_{k=1}^{\infty}$ in M

Such that $\|x+m_k\| \rightarrow 0$ as $k \rightarrow \infty$.

$$\Rightarrow \lim_{k \rightarrow \infty} m_k = -x$$

$\Rightarrow -x \in M$. (since M is closed and $\langle m_k \rangle$ is a sequence in M converging to $-x$) ——— (1)

$$\Rightarrow x \in M \quad (\because M \text{ is subspace})$$

$$\Rightarrow x+M = M \quad (\text{the zero element of } N/M)$$

thus we have show that

$$\|x+M\| = 0 \Rightarrow x+M = M \quad (\text{the zero of } N/M)$$

(n3): let $x+M, y+M \in N/M$. Then

$$\|(x+M) + (y+M)\| = \|(x+y)+M\|$$

by definition of addition of coset.

$$= \inf \{ \|x+y+m\| : m \in M \} \quad \text{————— (2)}$$

$$= \inf \{ \|x+y+m+m'\| : m \in M, m' \in M \} \quad \text{————— (3)}$$

[$\because M$ is subspace, the sets in (2) and (3) are

the same]

$$= \inf \{ \|x+m+(y+m')\| : m, m' \in M \}$$

$$\leq \inf \{ \|x+m\| + \|y+m'\| : m, m' \in M \}$$

using (n3) for N since $x+m, y+m' \in N$

$$= \inf \{ \|x+m\| : m \in M \} + \inf \{ \|y+m'\| : m' \in M \}$$

$$= \|x+M\| + \|y+M\|$$

(n4):

$$\|\alpha(x+M)\| = \inf \{ \|\alpha(x+m)\| : m \in M \}$$

$$= \inf \{ |\alpha| \|x+m\| : m \in M \}$$

by (n4) for N .

$$= |\alpha| \inf \{ \|x+m\| : m \in M \}$$

$$= |\alpha| \|x+M\|$$

Hence N/M is a normed linear space.

Now, we prove that if N is complete, then so is N/M . We know that a Cauchy sequence is convergent if and only if it has a convergent subsequence.

Let $\langle s_n + M \rangle_{n=1}^{\infty}$ be Cauchy sequence in N/M .

So that $s_n \in N$.

We construct a convergent subsequence of this sequence as follows:

Since $\langle s_n + M \rangle$ is a Cauchy sequence, for $\epsilon = \frac{1}{2}$ there exists a positive integer n_1 such that

$$n, m \geq n_1 \Rightarrow \| (s_n + M) - (s_m + M) \| < \frac{1}{2}$$

Set $s_{n_1} = x_1 \in N$.

Similarly for $\epsilon = (\frac{1}{2})^2$ there exists a positive integer $n_2 > n_1$ such that

$$n, m \geq n_2 \Rightarrow \| (s_n + M) - (s_m + M) \| < (\frac{1}{2})^2$$

Set $s_{n_2} = x_2 \in N$.

In general having chosen x_1, x_2, \dots, x_k and n_1, n_2, \dots, n_k . Let $n_{k+1} > n_k$ be such that

$$n, m \geq n_k \Rightarrow \| (s_n + M) - (s_m + M) \| < (\frac{1}{2})^{k+1}, \text{ set}$$

$x_{k+1} = s_{n_{k+1}}$.

Thus we have obtained a subsequence

$\langle x_k + M \rangle_{k=1}^{\infty}$ of $\langle s_n + M \rangle$

such that $\| (x_{k+1} + M) - (x_k + M) \| < (\frac{1}{2})^k$.

$k = 1, 2, 3, \dots$

It will now be shown that this subsequence

Converges to an element of N/M . We begin by

choosing any vector y_1 in $x_1 + M$,

After choosing y_1 , we select $y_2 \in x_2 + M$ such that

$$\|y_1 - y_2\| < \frac{1}{2}.$$

This is possible for $y_1 \in x_1 + M \Rightarrow y_1 = x_1 + m_1$ for

some $m_1 \in M$ and $\|(x_1 + M) - (x_2 + M)\| < \frac{1}{2} \Rightarrow$

$\inf \{ \|x_1 - x_2 + m\| : m \in M \} < \frac{1}{2} \Rightarrow \exists$ an $m_0 \in M$ such that

$$\|x_1 - x_2 + m_0\| < \frac{1}{2}.$$

$\Rightarrow \|y_1 - y_2\| < \frac{1}{2}$ where $y_2 = x_2 + m_0 + m_1 \in x_2 + M$. We

next select y_3 in $x_3 + M$ such that $\|y_2 - y_3\| < (\frac{1}{2})^2$.

Continuing in this manner we get a sequence

$\langle y_n \rangle$ in N such that

$$\|y_n - y_{n+1}\| < \left(\frac{1}{2}\right)^n \quad \text{--- (A)}$$

We claim that, $\langle y_n \rangle$ is a Cauchy sequence in N . For

if $\epsilon > 0$ be given, we can choose a positive integer m_0 so large that

$$\frac{1}{2^{m_0-1}} < \epsilon$$

then for $n > m \geq m_0$, we have

$$\|y_m - y_n\| = \| (y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \dots + (y_{n-1} - y_n) \| \leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \dots + \|y_{n-1} - y_n\|$$

$$\stackrel{(A)}{<} \sum_{i=m}^{n-1} \left(\frac{1}{2}\right)^i \quad \text{by (A)}$$

$$< \sum_{i=m}^{\infty} \left(\frac{1}{2}\right)^i = \left(\frac{1}{2}\right)^m + \left(\frac{1}{2}\right)^{m+1} + \dots$$

$$= \frac{\left(\frac{1}{2}\right)^m}{1 - \frac{1}{2}} = \frac{1}{2^{m-1}} \leq \frac{1}{2^{m_0-1}} < \epsilon$$

Hence $\langle y_n \rangle$ is a Cauchy sequence in N and

Since N is complete, there exists a vector y in N such that $y_n \rightarrow y$.

Finally we prove that

$x_{n+M} \rightarrow y+M \in N/M$, we have

$$\begin{aligned} \|(x_{n+M}) - (y+M)\| &= \|x_n - y + M\| \\ &= \inf \{ \|x_n - y + m\| : m \in M \} \\ &\leq \|x_{n+m} - y\| \text{ for all } m \in M. \end{aligned}$$

Since $y_n = x_{n+m_n}$ for some $m_n \in M$, we conclude that

$$\|(x_{n+M}) - (y+M)\| \leq \|y_n - y\| \rightarrow 0 \text{ since } y_n \rightarrow y$$

Hence $x_{n+M} \rightarrow y+M \in N/M$ and consequently N/M is complete.

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